Lecture 2:
Mirror descent and online decision making

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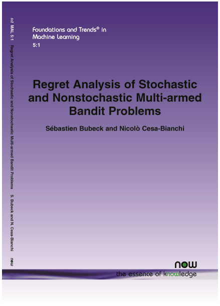

Stability as an algorithmic guiding principle

Recall that we are looking for a rule to select $p_t \in \Delta_n$ based on $\ell_1, \ldots, \ell_{t-1} \in [-1, 1]^n$, such that we can control the regret with respect to any comparator $q \in \Delta_n$:

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In other words p_{t+1} (which can depend on ℓ_t) is trading off being "good" for ℓ_t, while at the same time remaining close to p_t.

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Connection: If i_t is played at random from p_t, and consequent samplings are appropriately coupled, then the term we want to bound

$$\sum_{t=1}^{T} \langle \ell_t, p_{t+1} - q \rangle + \sum_{t=1}^{T} \|p_t - p_{t+1}\|_1 ,$$

exactly corresponds to the sum of expected service cost and expected movement when the metric is trivial (i.e., $d \equiv 1$).

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Side comment: another equivalent definition is as follows, say with $x_1 = 0$,

$$x_{t+1} = \operatorname*{argmin}_{x \in \mathbb{R}^n} \langle x, \sum_{s \leq t} \ell_s \rangle + \frac{1}{2\eta} \|x\|_2^2 \, .$$

This view is called "Follow The Regularized Leader" (FTRL)

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Mirror map/regularizer: convex function $\Phi : \mathcal{D} \supset K \to \mathbb{R}$.
Bregman divergence: $D_\Phi(x; y) = \Phi(x) - \Phi(y) - \nabla\Phi(y) \cdot (x - y)$.
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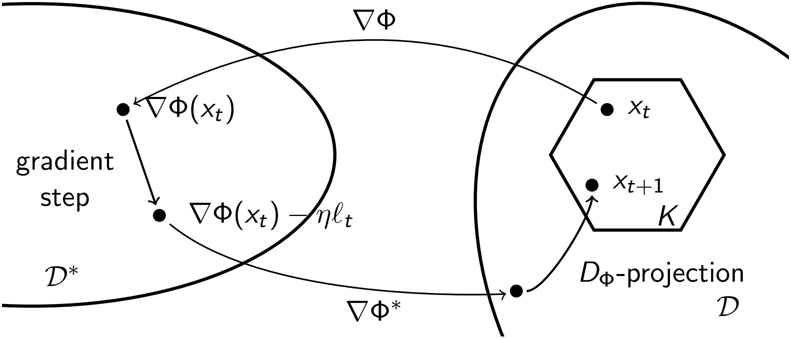

Continuous-time mirror descent

Assume now a continuous time setting where the losses are revealed incrementally and the algorithm can respond instantaneously: the service cost is now $\int_{t\in\mathbb{R}_+} \ell(t)\cdot x(t)dt$ and the movement cost is $\int_{t\in\mathbb{R}_+} \|x'(t)\|_1 dt$.

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$$\Leftrightarrow \nabla\Phi(x(t+\varepsilon)) - \nabla\Phi(x(t)) + \varepsilon\eta\ell(t) \in -N_K(x(t+\varepsilon))$$

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$$\begin{aligned} x(t+\varepsilon) &= \operatorname*{argmin}_{x \in K} D_\Phi(x, \nabla \Phi^*(\nabla \Phi(x(t)) - \varepsilon \eta \ell(t))) \\ &\Leftrightarrow \nabla \Phi(x(t+\varepsilon)) - \nabla \Phi(x(t)) + \varepsilon \eta \ell(t) \in -N_K(x(t+\varepsilon)) \\ &\Leftrightarrow \nabla^2 \Phi(x(t)) x'(t) \in -\eta \ell(t) - N_K(x(t)) \end{aligned}$$

Theorem (BCLLM17)

The above differential inclusion admits a (unique) solution $x : \mathbb{R}_+ \to \mathcal{X}$ provided that K is a compact convex set, Φ is strongly convex, and $\nabla^2 \Phi$ and ℓ are Lipschitz.

The basic calculation

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&= (\eta\ell(t)+\lambda(t))\cdot(y-x(t)) \\
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Lemma

The mirror descent path $(x(t))_{t\geq 0}$ satisfies for any comparator point y,

$$\int \ell(t) \cdot (x(t) - y)dt \leq \frac{D_\Phi(y; x(0))}{\eta}.$$

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Thus to control the regret it only remains to bound the movement cost $\int_{t \in \mathbb{R}_+} \|x'(t)\|_1 dt$ (recall that this continuous time setting is only valid for the 1-lookahead setting, i.e., MTS).

Controlling the movement and how the entropy arises

How to control $\|x'(t)\|_1 = \|(\nabla^2 \Phi(x(t)))^{-1}(\eta \ell(t) + \lambda(t))\|_1$? A particularly pleasant inequality would be to relate this to say $\eta \ell(t) \cdot x(t)$, in which case one would get a final regret bound of the form (up to a multiplicative factor $1/(1-\eta)$):

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Ignore for a moment the Lagrange multiplier $\lambda(t)$ and assume that $\Phi(x) = \sum_{i=1}^n \varphi(x_i)$. We want to relate $\sum_{i=1}^n \ell_i(t)/\varphi''(x_i(t))$ to $\sum_{i=1}^n \ell_i(t) x_i(t)$.

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In particular $\|x'(t)\|_1 \leq 2\eta \ell(t) \cdot x(t)$.

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We note that this algorithm is exactly a continuous time version of the MW studied at the beginning of the first lecture.

The more classical discrete-time algorithm and analysis

Ignoring the Lagrangian and assuming $\ell'(t) = 0$ one has

$$\partial_t^2 D_\Phi(y; x(t)) = \nabla^2 \Phi(x(t))[x'(t), x'(t)] = \eta^2 (\nabla^2 \Phi(x(t)))^{-1}[\ell(t), \ell(t)] \,.$$

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Thus provided that the Hessian of Φ is well-conditioned on the scale of a mirror step, one expects a discrete time analysis to give a regret bound of the form (with the notation $\|h\|_x = \sqrt{\nabla^2 \Phi(x)[h, h]}$)

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Theorem

The above is valid with a factor $2/c$ on the second term, provided that the following implication holds true for any $y_t \in \mathbb{R}^n$,

$$\nabla \Phi(y_t) \in [\nabla \Phi(x_t), \nabla \Phi(x_t) - \eta \ell_t] \Rightarrow \nabla^2 \Phi(y_t) \succeq c \nabla^2 \Phi(x_t) \,.$$

For FTRL one instead needs this for any $y_t \in [x_t, x_{t+1}]$.

MW is mirror descent with the negentropy

Let $\Phi(x) = \sum_{i=1}^{n}(x_i \log x_i - x_i)$ and $K = \Delta_n$. One has $\nabla\Phi(x) = \log(x_i)$ and thus the update step in the dual looks like:

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Note also that the well-conditioning comes for free when $\ell_t(i) \geq 0$, and in general one just needs $\|\eta \ell_t\|_\infty$ to be $O(1)$.

Propensity score for the bandit game

Key idea: replace ℓ_t by $\widetilde{\ell}_t$ such that $\mathbb{E}_{i_t \sim p_t} \widetilde{\ell}_t = \ell_t$. The propensity score normalized estimator is defined by:

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The Exp3 strategy corresponds to doing MW with those estimators. Its regret is upper bounded by,

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where $\|h\|_{p,*}^2 = \sum_{i=1}^n p(i) h(i)^2$. Amazingly the variance term is automatically controlled:

$$\mathbb{E}_{i_t \sim p_t} \sum_{i=1}^{n} p_t(i) \widetilde{\ell}_t(i)^2 \leq \mathbb{E}_{i_t \sim p_t} \sum_{i=1}^{n} \frac{\mathbb{1}\{i = i_t\}}{p_t(i_t)} = n.$$

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$$\widetilde{\ell}_t(i) = \frac{\ell_t(i_t)}{p_t(i)} \mathbb{1}\{i = i_t\} \, .$$

The Exp3 strategy corresponds to doing MW with those estimators. Its regret is upper bounded by,

$$\mathbb{E} \sum_{t=1}^{T} \langle p_t - q, \ell_t \rangle = \mathbb{E} \sum_{t=1}^{T} \langle p_t - q, \widetilde{\ell}_t \rangle \leq \frac{\log(n)}{\eta} + \eta \mathbb{E} \sum_{t} \|\widetilde{\ell}_t\|_{p_t,*}^2 \, ,$$

where $\|h\|_{p,*}^2 = \sum_{i=1}^n p(i) h(i)^2$. Amazingly the variance term is automatically controlled:

$$\mathbb{E}_{i_t \sim p_t} \sum_{i=1}^{n} p_t(i) \widetilde{\ell}_t(i)^2 \leq \mathbb{E}_{i_t \sim p_t} \sum_{i=1}^{n} \frac{\mathbb{1}\{i = i_t\}}{p_t(i_t)} = n.$$

Thus with $\eta = \sqrt{n \log(n)/T}$ one gets $R_T \leq 2\sqrt{T n \log(n)}$.

Simple extensions

- Removing the extraneous $\sqrt{\log(n)}$
- Contextual bandit
- Bandit with side information
- Different scaling per actions

More subtle refinements

- Sparse bandit
- Variance bounds
- First order bounds
- Best of both worlds
- Impossibility of $\sqrt{T}$ with switching cost
- Impossibility of oracle models
- Knapsack bandits